Recursive definitions are familiar in mathematics. For instance, the function $f$ defined by

\[
\begin{align*}
f(0) &= 1, \\
f(1) &= 1, \\
f(x + 2) &= f(x + 1) + f(x),
\end{align*}
\]
gives the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, \ldots. (The study of difference equations concerns the problem of going from recursive definitions to algebraic definitions. The Fibonacci sequence is give by the algebraic definition

\[
f(x) = \frac{\sqrt{5}}{5} \left( \frac{1 + \sqrt{5}}{2} \right)^{x+1} - \frac{\sqrt{5}}{5} \left( \frac{1 - \sqrt{5}}{2} \right)^{x+1}.
\]

The primitive recursive functions are an example of a broad and interesting class of functions that can be obtained by such a formal characterization. **Definition** The class of primitive recursive functions is the smallest class $C$ (i.e., intersection of all classes $C$) of functions such that

i. All constant functions, $\lambda x_1 x_2 \cdots x_k[m]$ are in $C$, $1 \leq k$, $0 \leq m$;

ii. The successor function, $\lambda x[x + 1]$, is in $C$;

iii. All identity functions, $\lambda x_1 \cdots x_k[x_i]$ are in $C$, $1 \leq i \leq k$;

iv. If $f$ is a function of $k$ variables in $C$, and $g_1, g_2, \ldots, g_k$ are (each) functions of $m$ variables in $C$, then the function $\lambda x_1 \cdots x_m[f(g_1(x_1, \ldots, x_m), \ldots, g_k(x_1, \ldots, x_m))]$ is in $C$, $1 \leq k, m$;

v. If $h$ is a function of $k + 1$ variables in $C$, and $g$ is a function of $k - 1$ variables in $C$, then the unique function $f$ of $k$ variables satisfying

\[
\begin{align*}
f(0, x_2, \ldots, x_k) &= g(x_2, \ldots, x_k), \\
f(y + 1, x_2, \ldots, x_k) &= h(y, f(y, x_2, \ldots, x_k), x_2, \ldots, x_k)
\end{align*}
\]
is in $C$, $1 \leq k$. (For (v), “function of zero variables in $C$” is taken to mean a fixed integer.)